POLAR REPRESENTATION OF THE PERCEPTION OF HARMONIC STRUCTURES IN MUSIC

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Abstract: The dissonance level of intervals are represented in polar coordinates over the twelve notes of an octave. This representation shows a recognizable pattern of the dissonance of the intervals with respect to the fundamental note at the origin (the pole). The polar coordinates are applied to visualize modes and by the application of superposition of the effects, to the chords with the assumption of the pole of the diagram as modal or bass notes. A Tonal Space, built with the fundamental tonal triad as pole, shows the tension of the notes and chords with respect to the tonic. The visualization of the perception of musical structures is achieved in a simple way.
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1 Introduction

The relationship between music and mathematics or geometry is very strong and typically the second tries to model the first one. The motivation in this research, as a passionate applied scientist and apprentice musician, follows the need and curiosity for a model that is able to show, in a simple way, the sound quality and functional aspects of harmony that the musicians talks about every day. I didn't find such type of representation among the plethora of the models that are already existing,
What follows is an exploration trip starting from a simple mathematical and geometrical (polar coordinates) representation applied to a fundamental psychoacoustic quality of consonance and dissonance of the intervals of the twelve notes. These qualities are substantially subjective but have psychoacoustic common basis that have been analyzed and modeled by many authors.
The different dissonance of the musical structures, for instance the chords, may play different roles when considered by themselves or in a tonal context. That aspect has been modeled by a Dual-Process theory which considers the sound perception as decomposed in sensory and tonal dissonances [Johnson-Laird2012]. In this work the mathematical representation is consequently adapted to provide a reference that highlight the pure dissonance of the isolated structures and, from the other side, the tension within a tonal reference.
The presentation of the results begins with the polar representation of the intervals in the intervals space. The modes and chords are shown in this space. Then the Tonal Space is defined adopting the tonal triad as reference pole for the polar diagrams.

2 The Interval Space

Harmony begins from two notes, and the acoustic perception quality intervals of two notes is the building block of all the exploration. The perception quality is defined in terms of consonance or dissonance. The dissonance of an interval is an elusive quality that involves psychophysical and apparently cultural aspects. Many theories have been developed in the last centuries to give an explanation and to provide a model: these started from the fundamental works of Helmholtz [HH1877], based on the wave partials to its contemporary developments and more recent theories that deviates from this purely mathematical approach on the harmonics and considers the brain models and related measures.

A scale of the dissonance from pure acoustic experience have been stated by some authors. The Hindemith historical dissonance level scale [PH1937] was largely influential; subsequently other authors have shown different scales including results from experimental measures. Experimental tests are reported in the publication [N&H1988].
The representation of the interval dissonance in a polar way is where this story begins.

2.1 Polar representation

The representation of the degree of dissonance of the intervals is made by a polar diagram to visualize them, in whichever way they are determined.
In a polar diagram, a bi-dimensional vector is represented by the coordinates: the absolute value and the angle with respect to a reference orientation. The vector can be represented as an arrow from the origin to the coordinated point. The dissonance of an interval is considered as a mathematical vector of the distance between a note and the note considered as the origin where the length of the vector is the dissonance degree. The circular domain is divided into 12 points as the number of semitones in an octave.
The polar representation of an interval is shown in figure 1.

The reference orientation is vertical up (12 o'clock) to show better the symmetric qualities that are somehow expected. Infact the intervals on the left side are the inversions of the right side ones. For instance a 2 interval is a m7 inverted interval.
In a written form the coordinates of the dissonance of intervals can be represented as follows:

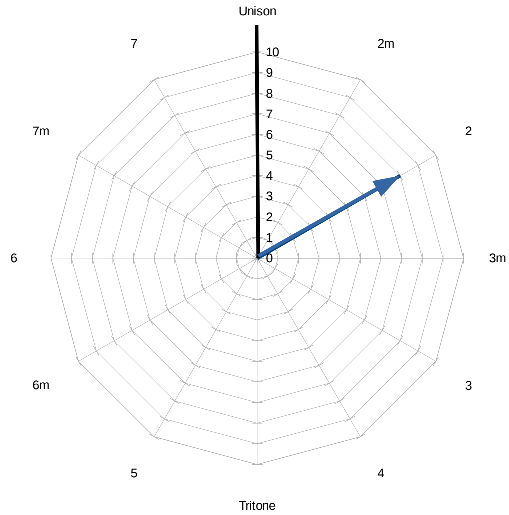


Figure 1. Example of an 2nd interval dissonance in polar coordinates. The reference note is the pole. The vector amplitude represents the dissonance level.

$$d_i=(L,i) ;$$

where: $d_i \in R$ is the dissonance vector for the interval "i", $L \in Y$, Y={ 0,...max level of dissonance} are the dissonance level and $i \in I$ I={Unison, m2, 2, m3, 3, 4, aug4 or tritone, 5, m6, 6, m7, 7} the intervals set in one octave, at the corresponding angles {0°,30°,60°,...330°}.
The Interval Space is the coordinate space where the vectors are defined in terms of the above defined interval domain.

2.2 Dissonance levels in polar representation

Models and measures of interval dissonance can be represented in polar coordinates.
The level of dissonance can be considered as a function S_I: which collects the values of dissonance for all the interval of the domain, specifically the octave in frequency.

$$D=S_I(i) \text{ for } i \in I \text{ , I=\{Unison, m2, 2, m3, 3, 4, tritone, 5, m6, 6, m7, 7\}.}$$

The representation of the dissonance functions, as polar diagrams, of the present results on the dissonance values, is as considered as the base for the subsequent development.
The classical scale of the harmonic force in Hindemith [PH1937], is represented in fig.2 where a lower harmonic force is interpreted as higher dissonance. The tritone values is not defined because it turns out to be variable in function of the context it is used. By comparison another example of the dissonance classification, which is presumably closer to a modern and jazz taste, is the one reported by Dan Hearle [DH1980].

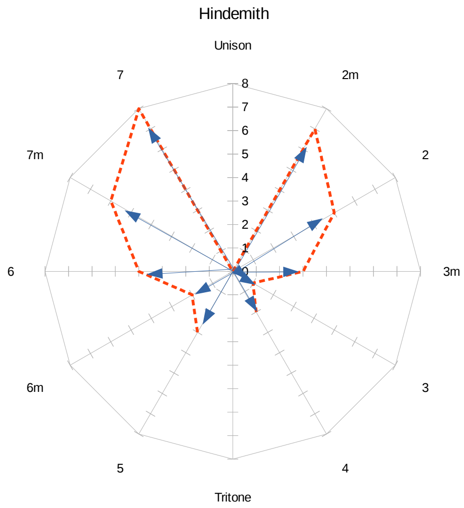


Figure 2: Hindemith interval dissonance levels. The polar function is shown by the dashed line.

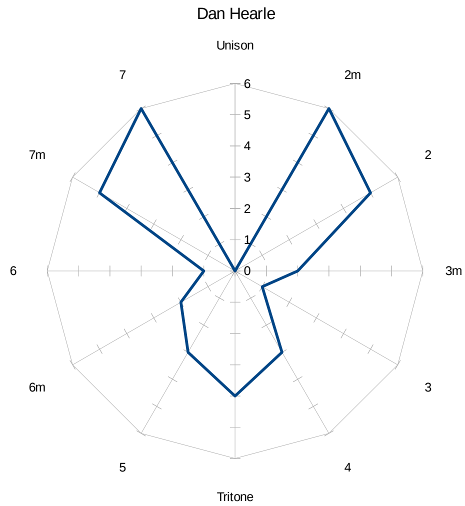


Figure 3: The polar diagram of the dissonant indexes of the intervals according

In figure 4, the dissonance levels are taken from one of the theories that derives from the partial which has been originated by the Helmholtz and subsequently revised by several authors. This approach, seems to reproduce with some degree of accuracy the values that comes from experiments with listeners. The data are taken from the diagram in [Cook 2017] for the case with six partials.

In this study the reference data are taken directly for an experiment in order to avoid to introduce modeling errors.

The experimental data of [N&F1988] have been considered as reference in the following part of this work. The values are:

Interval	Dissonance
Unison	0
2m	5.7
2	3.9
3m	2.6
3	2
4	2
Tritone	4
5	1.7
6m	3
6	2.4
7m	3.3
7	5.3

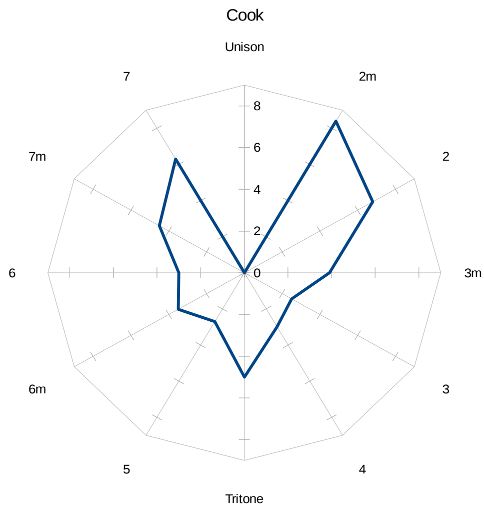


Figure 4: The model from [Cook 2017].

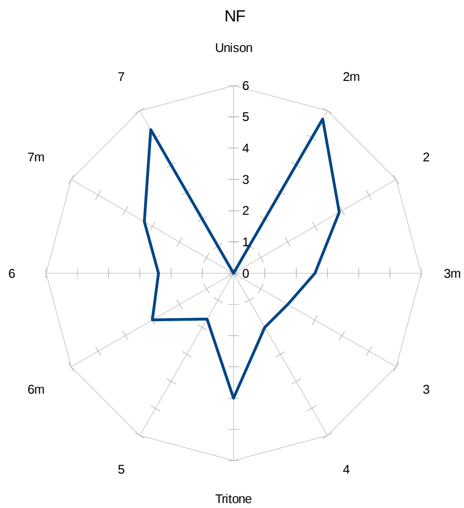


Figure 5: The interval dissonance level from [N&F1988] is considered as the reference one.

All the level data I found have strong similarities in shape. The relative amplitude of the dissonance levels can be different.

2.3 Scales and modes in the Interval Space

A first application of the interval space polar plots is for scales and modes. Scales and modes are a subset of the twelve notes which are referred to a specific modal note.
For a generic mode, the set of dissonance is defined as:

$$D=S_I(i) \text{ for } \quad i \in M \quad \text{where}$$

M is the subset of the intervals of the specific mode.
For instance in the major scale (ionian mode), M={Unison, 2, 3, 4, 5, 6, 7} is represented in figure 6.

The representation of minor natural and the minor harmonic scales, in figure 7, show the difference in the dissonance levels.

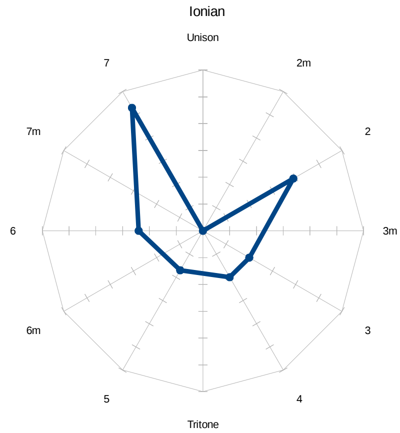
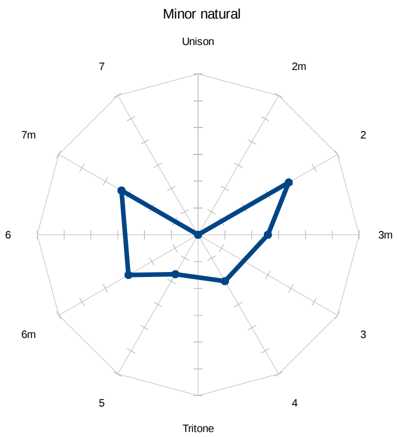


Figure 6: The major scale visualized.



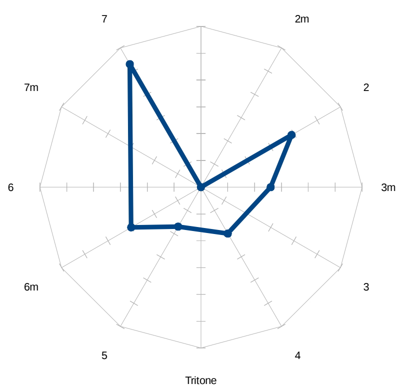


Figure 7:
The minor natural and minor harmonic scales. The pole is the modal reference note.

From the diagrams reported in Appendix A it is visible for the major diatonic modes, the different degree of dissonance as somehow related to the pattern dimension. A more precise definnition of the metric is part of subsequent study extensions.
It is a common and shared perception to associate a “color” to each mode in accordance to a scale degree from the Lydian mode as the brightest to the Locrian as the darkest:

Lydian – Ionian – Mixolydian – Dorian – Eolian – Phrygian - Locrian.
< bright dark>

This color degree scale is visible in the patterns with the brightest being identified by a large “left” component (associated to a 7th interval) and darker with a more significant “right” one (m7 interval).
The symmetric scales, like the exatonic or the diminished ones, are reproduced equally from any of their noteso that they can be represented as referred to one of its note considered as reference.

2.4 Chords in the Interval Space

The polar model applied to the intervals is also a way to visualize the perceived dissonance, or sonority as defined by some authors, of chords.
A chord is composed by intervals with respect to the bass note and the internal intervals among the other chord notes. A natural way to represent a chord on the interval space is therefore to plot the intervals belonging to the chord.
In a basic interval effect, only the interval from the reference note within the octave can be considered.
In a more refined representation the internal intervals are included by superimposing the interval vectors for each subsequent internal note.
The base hypothesis is that the resulting sonority effect is given by the superposition of the effects of the individual intervals. This is a simple and natural assumption that it is hoped to be valid at least for the major results.

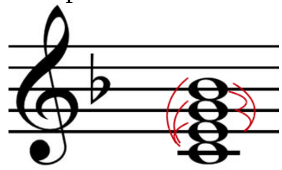


Figure 8: A four note chord is composed by 6 interval, example on the C^7 chord.

The superposition is straightforward for intervals which are distinct: the superposition is the union of the vectors.
The process for a generic X^7 chord (where X stands for a generic tone) is shown in figure 8.

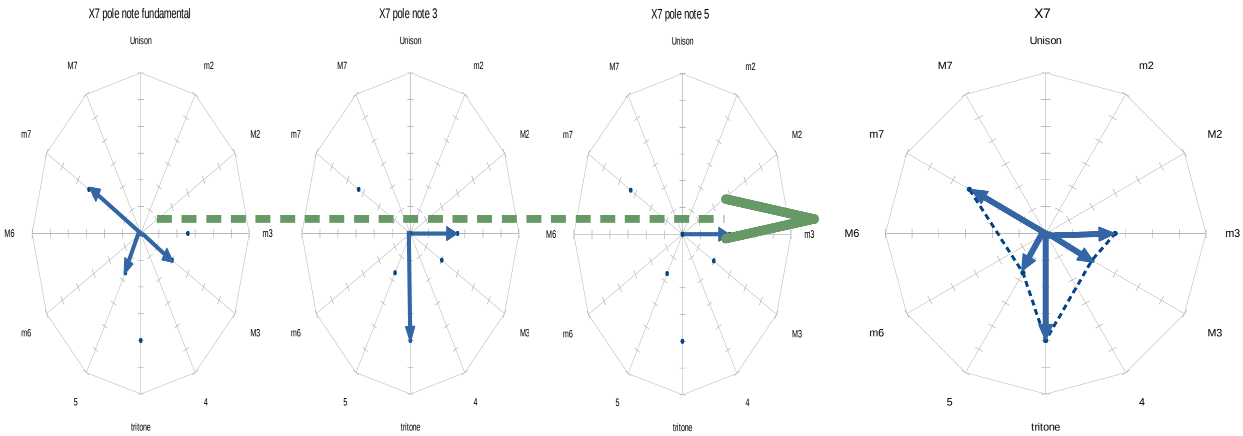


Figure 9: superposition of the intervals for a X^7 chord . The resulting polar diagram includes all the intervals. The pole of the diagram is the set of the notes of the chord except the highest.

The result for the X^7 chord is shown in figure 10.

From a formal point of view:

$$D_c = \bigcup_{i \in C} d_i \quad \text{where C=\{intervals of the chord\}.}$$

By considering the union of vector the intervals that are duplicated do not have effect.

The superposition can alternatively consider to sum the effects of duplicated intervals. The effect of duplicated intervals in the triads and the case when they they are adjacent has to be evaluated. Some insight for the triads can be derived by the experiments represented in Table1 of [Cook2009].
This effect is not discussed here but, for the sake of generality and to take into account this effect, the effect of duplicated intervals can be modeled by a weighted sum of the intervl effect.

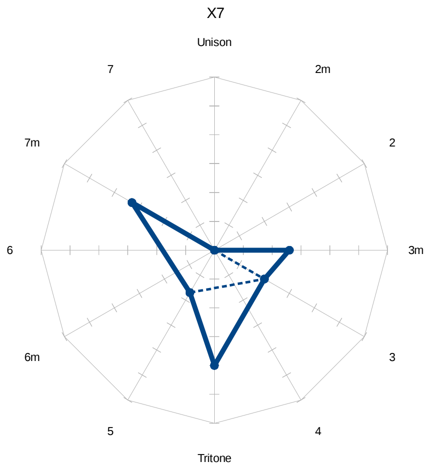


Figure 10: The X^7 chord represented. The dashed line connects only the fundamental intervals.

The diagrams for several chord types reported in Appendix B, visualize the quality of the chords in terms of dissonance. Both the basic and the completed intervals are shown in the same diagrams. Also in this case a correspondence between the expected chord dissonance and the related pattern sizes can be found. A deviation with the experiments is visible for the diminished and augmented triads whose relative dissonance doesn't appear confirmed by the above mentioned experiments.

3 The Tonal Space

The application of the above polar representation is referred to notes and chords as isolated elements. In a tonal context things may change. One of such effects is the dual behavior of the 4^{th} interval in different musical context, as described in music theory manuals.

But how to consider the intervals in the dual tonal context?

According to a Dual-Process Theory of dissonance [Johnson-Laird2012], originated by Helmholtz and then in [T1984], the tonal dissonances may refer to a separate cognitive process with respect to the isolated sensory ones. In addition chords that are consistent with major triad are more consonant.

From this theory the hypothesis is made that, in tonal context, chords which are further away from the tonic triad are more dissonant, or have higher tension, with respect to the tonality.

The distance of a note from the tonal triad is therefore considered as the value of tension. The term "tension" is here preferred to "dissonance" being a measure of tendency towards the tonality reference.

A Tonal Space is defined as the sets of the scale degrees and related tension vector according to the relation:

$$t = S_T(x)$$

$$\text{for} \quad t \in T \quad T = R \text{ value of tension;} \quad x \in X \quad X = \{\text{scale degree}\},$$

3.1 Major Tonal Space

As in the case of the chords in the interval space, the function S_T representing the tension for each tone of the scale is considered a linear effect superposition. Therefore it is computed as the sum of the dissonances, defined in the Interval Space, of the three notes of the tonic triad. In formal expression the tension T in the Tonal Space is expressed as:

$$t = S_T(x) = a_1\, S_I(i) + a_3\, S_I(i_3) + a_5\, S_I(i_5) \quad \text{for} \quad i \in I \quad , \quad i_3 \in I \rightarrow 3 \quad , \quad i_5 \in I \rightarrow 5$$

where, with some freedom from a rigorous formalism, $I \rightarrow n$ is intended the rotational shift of the I domain of a interval n from the tonic.

The coefficients a_1, a_3, a_5 can be used to introduce a weight in the three component contributions. In this paper it is considered $a_1 = a_3 = a_5 = 1$.

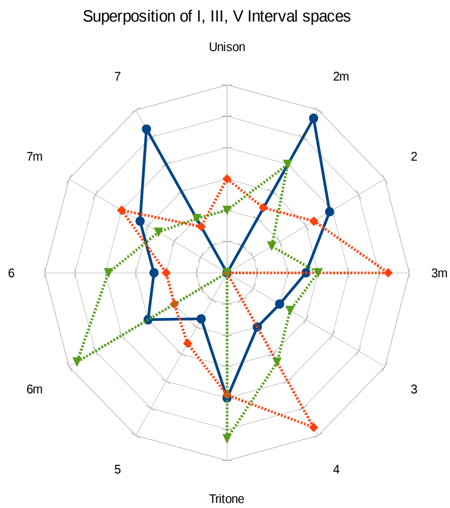


Figure 11: The rotated Interval Spaces of the three tonic chord notes

The first step of the process, the rotational shift, is depicted in Figure 11: the dissonance values are then added for each note degree.

The resulting Tonal Space is shown in figure 12, representing the tension of the notes in an octave with respect to the tone. The major diatonic scale is shown in figure 13 as subset of the diatonic notes of the whole Tonal Space.

$$t = S_T(x) \text{ for} \quad t \in T \quad T = R \text{ value of tension;} \quad x \in X \quad X = \{\text{diatonic tones}\}.$$

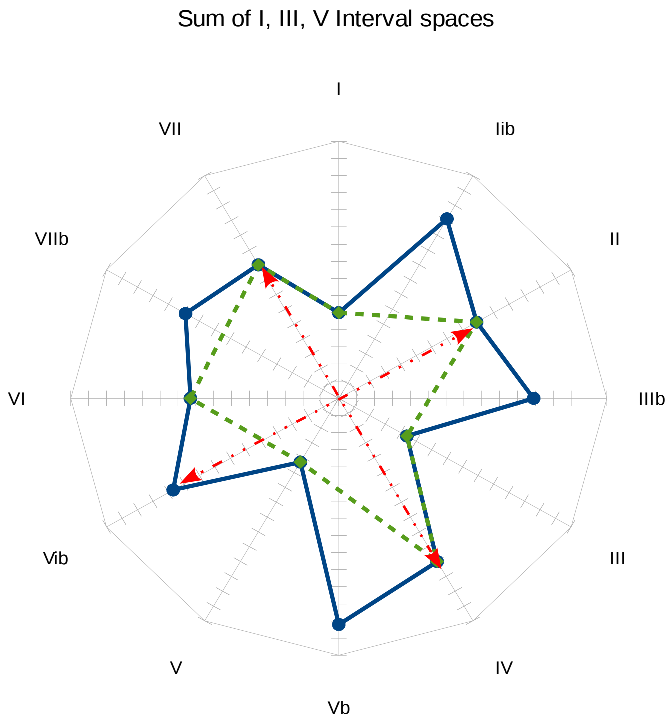


Figure 12: The Major Tonal Space. Main tritones in dashed red and diatonic notes in dashed green.

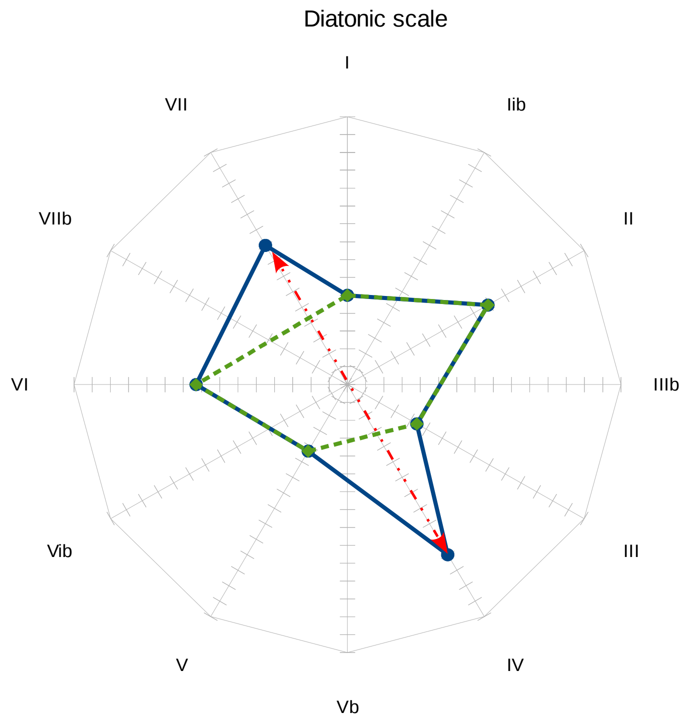


Figure 13: The diatonic scale in the Major Tonal Space. Tritone in dashed red and major pentatonic scale in dashed green.

From these two diagrams it is evident the role of the 4^{th} being termed “avoid” note in a tonal context by basically all the jazz theory books, e.g. [JMC2000].
Another aspect is shown by the notes on IV and VII degrees that have the strongest tension and are associated to the two dominant chords: the V^7 and VII° . These two notes create, by chance from this point of view, a tritone.

It is also interesting to plot the minor natural scale over the major tonal scale. The plot shows that chord from a minor scale which contain the II and VIb, e.g. $II^{\varnothing}$, act as external dominant, even if with a lower tension.

Further investigations can be done considering different scales and groups of notes, as shall be done later on with the chords.

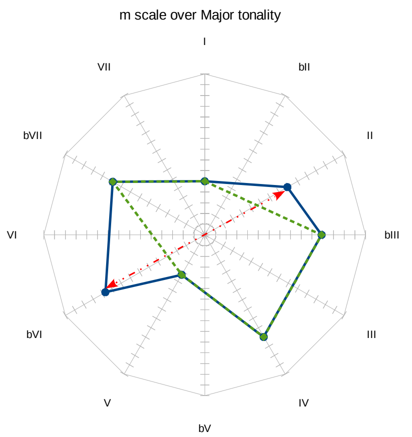


Figure 14: Minor natural scale in the Tonal space

3.2 Minor Tonal Space

What has been done for the major Tonal Space is replicated for the minor triad.
In formal expression the tension T in the Tonal Space is expressed as:

$$t=S_{Tm}(x)= S(i) + S(i_{m3}) +S(i_5)$$

for T=R value of tension; $x \in X$ X={diatonic tones}, $i \in I$, $i_{m3} \in I \rightarrow m3$, $i_5 \in I \rightarrow 5$

with the same meaning for $I \rightarrow n$ is intended the rotation shift of the I domain of a interval n from the tonic.

The result is shown in figure 15, representing the tension of the notes with respect to the minor fundamental triad. The tones Vib, VI, VIIb, VI that belong to the different minor modes (natural, harmonic, melodic) show similar tension values. The minor natural diatonic scale is shown in Figure 16, as subset of the diatonic notes of the whole Tonal Space.

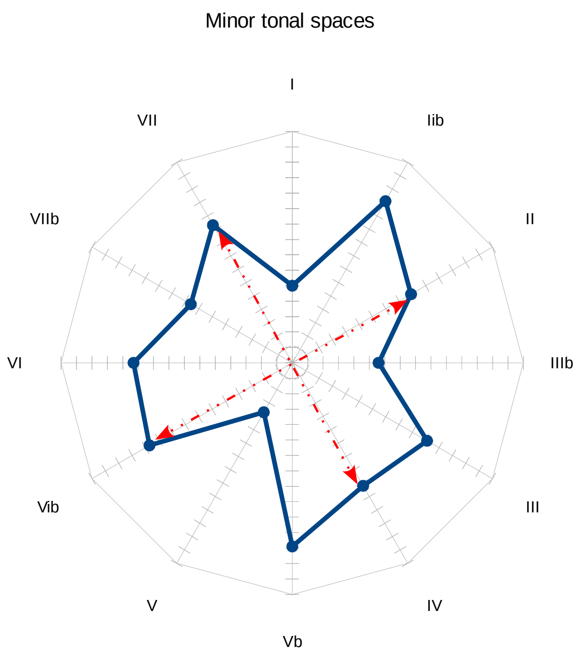


Figure 15: Minor Tonal Space. Main tritones in dashed red

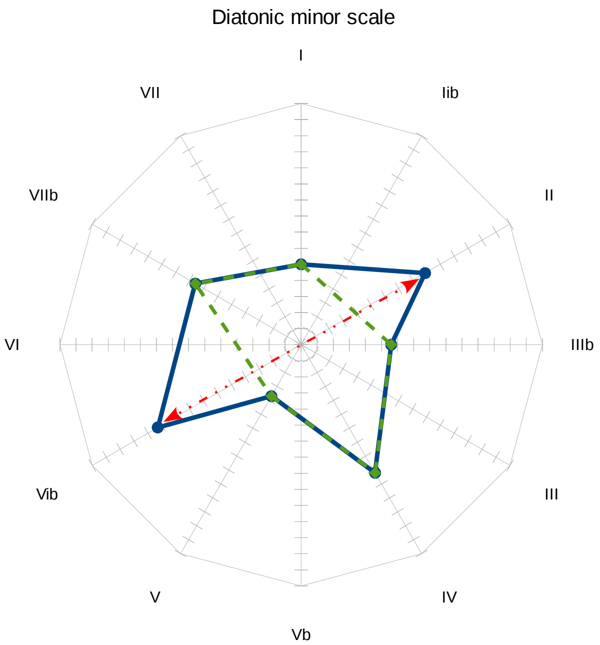


Figure 16: Minor natural space. Green dashed line for pentatonic scale and red vector for the tritone

Also in this case the two notes with the highest tension, (II and VIb) are a tritone away which is, from a geometrical point of view, orthogonal to the tritone of the major Tonal Space. This correspond to a m3 shift and in fact it is the same tritone of the relative major scale. This fact suggests that the dominant chords in the minor scale contains the two notes II and VIb. These chords are $VIIb^{7}$ and the II° (with all the diminished chords a 3m away). It is well known instead that the V^{7b9} is used as dominant chord associated to the harmonic minor scale.

3.3 Modes in Tonal Space

The modal perception within the tonal context is a subtle subject. It is a common practice in Jazz to use modes for melodies over tonal chords with a main goal to provide colors, or sonorities. All the mode belonging to the scale have all the same notes and nothing more can be extracted by the polar diagrams since the share all the same one: shown in figure 12 for the major tonality and 16 for the minor. Modes external to the tonality related to diatonic chords, e.g. C Lydian over $C^{\Delta 7}$ chord, have a different pattern in the Tonal Space.

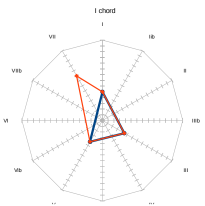
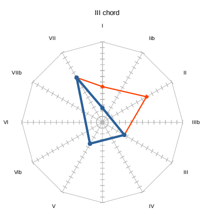
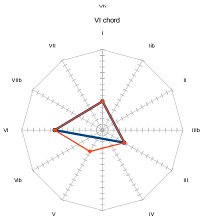
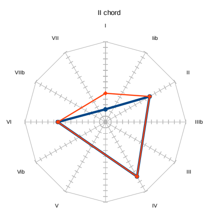
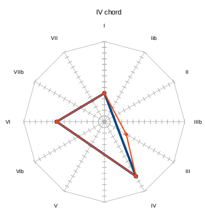
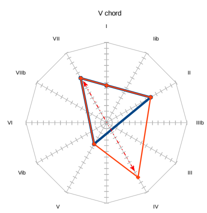
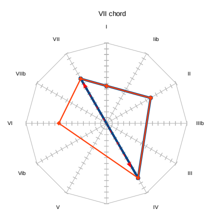
3.4 Chords in the Tonal Space

The tension of chords in the Tonal Space is straightforward once the Tonal Space is defined. A pattern is achieved by selecting the subset of notes of the chord. The result is related to the tension to the tonality while the intrinsic dual perceptive aspect of dissonance is visualized in the Interval Space.
From a formal definition the tension of a chord is expressed by the vector set:

$$T=S_T(c) \quad \text{for} \quad c \in D$$

where c are the chord tone and D is the diatonic set.
A set of diagrams is shown in Appendix C.

The traditional role of the chords functional harmony can be compared with the chord patterns in the next table. The different size of the patterns are approximatively associated to the expected tension in the groups.

Tonic	Subdominant	Dominant
  	 	 

4 Extensions

The dual approach separates two sensory aspects for analysis purposes. The combined effect of the sensory and tonal dissonance of the chords plays a role that is not here considered. For instance a C^7 chord has a double representation in the Interval and Tonal spaces.
Would this representation have a sense, it could be developed further to analyze the correspondence of the patterns with musical structures. The following topics are suggested.

Vector space
To develop the mathematics of the vectors associated to the intervals and metrics. Is there any musical sense in vector operators defining a metric space in mathematical sense?

Extensions to notes over the octave
The representation is limited to the octave. Due to the cyclic nature of the notes in the equal temperament, a sort of 3D continuous helicoid or spiral can be considered. The Interval Space in the polar circle would be the projection of the first octave on the base plane. In addition, since the dissonance decreased with the distance from the pole, it can be made a conjecture that the 3D Interval Space is a conical or logarithmic spiral like the one shown in figure 17.

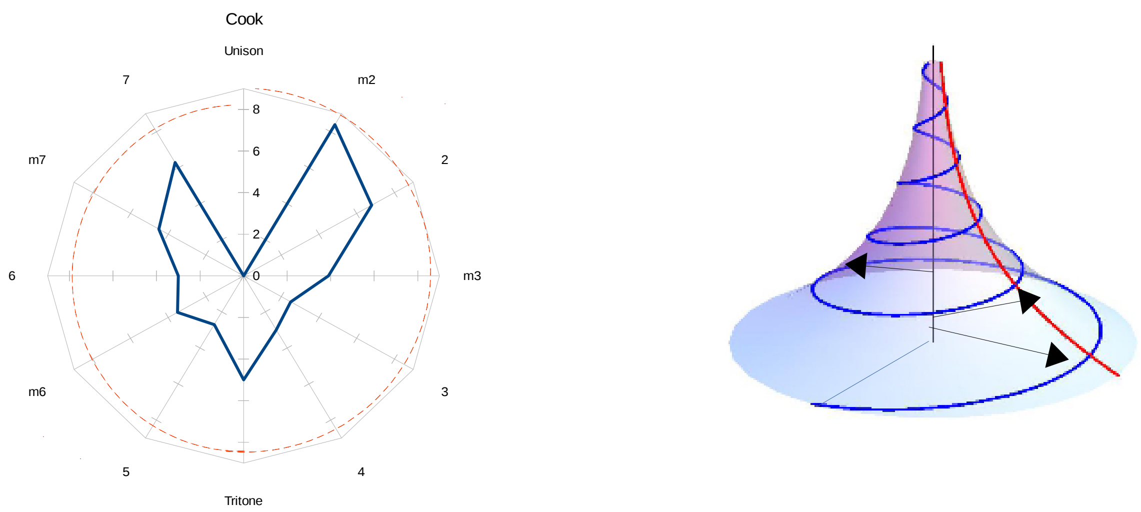


Figure 17: the spiral of intervals. The dissonance effect is decreasing with the interval amplitude. .
Figure from "Spirals on surfaces of revolution" by Cristian Lazureanu

5 Conclusions

The dissonance of intervals is considered as building block for a simple description of the perception of structures, modes, scales, chords, in harmony. Given a measure of the dissonance levels perceived for each interval, it's possible to obtain a nice pattern of the sound roughness. The basic idea is to transpose the dissonance and pitch into a polar coordinate with respect to a polar reference.
This formal representation can be applied to show properties of scales, chords in sensory dissonance on. The approach to model the harmonic structures is based on the superposition of the effects. In order to extend the model to the tonal context, the tonal triad is taken as polar reference. The distance, in terms of dissonance, of the twelve tones from the tonal triad gives the representation of the perceived Tonal Space.
The approach is quite simple and also the results are addressed to the basic aspects of the harmony in easy way. The obtained representations provide a fair correspondence in the qualities of perception at a first glance. Once validate by experiment data it could be useful in helping to visualize harmonic relationships in music.

Art in its most primitive state is a simple imitation of nature. But it quickly becomes imitation of nature in the wider sense of this idea, that is, not merely imitation of outer but also of inner nature. ...In its most advanced state, art is exclusively concerned with the representation of inner nature. (A.Shoenberg [AS:III-Consonance and dissonance])

6 References

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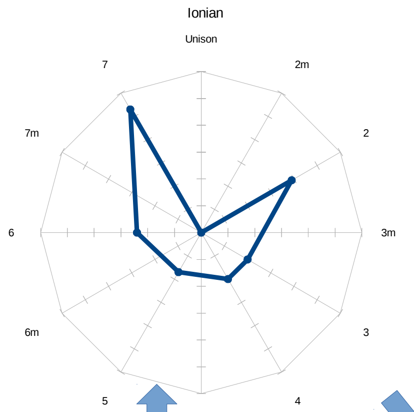
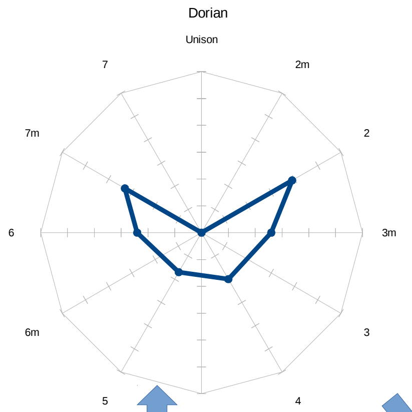
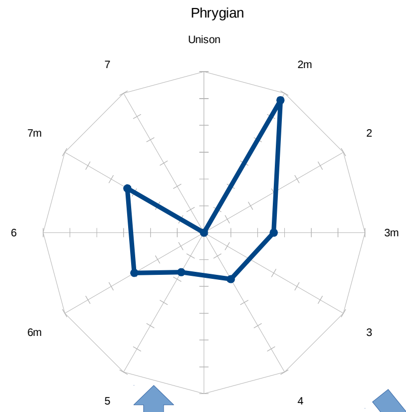
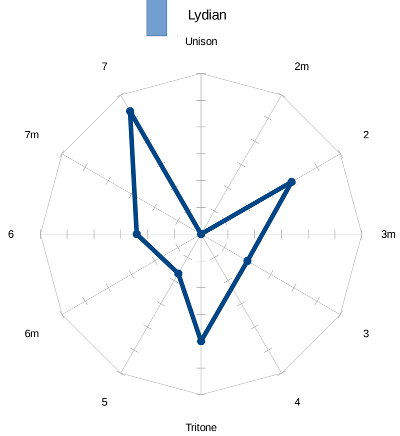
[VP]V.Persichetti: *Twentieth Century Harmony.* Norton&Company

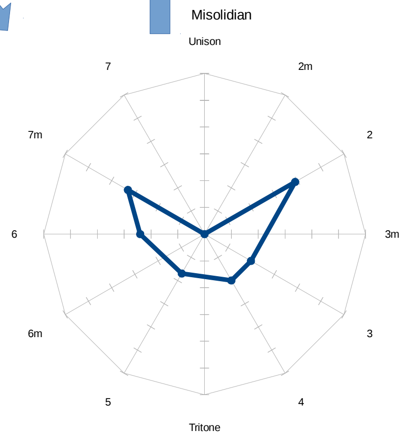
APPENDIX A: Modes

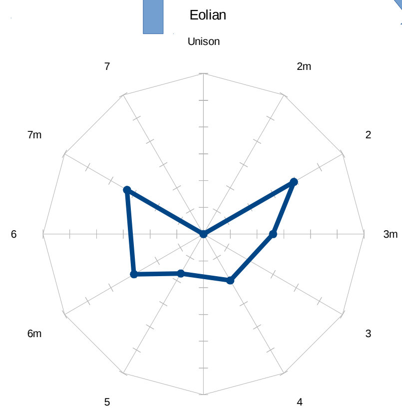






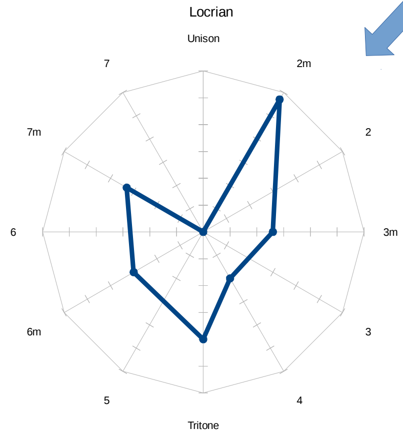




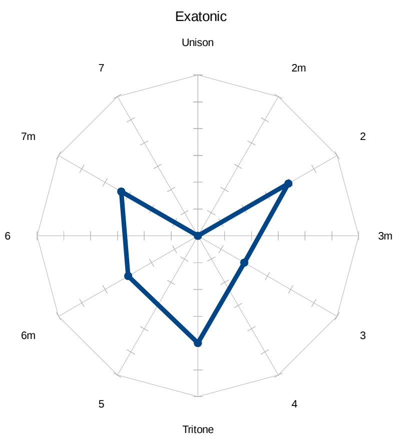


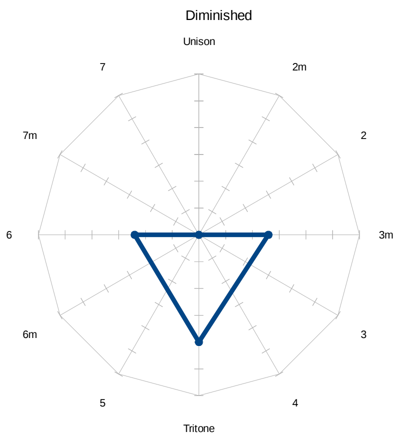
Brigth

Intermediate



NB: Periodic table of the modes. The arrows track the “color” from the brightest to the darkest.

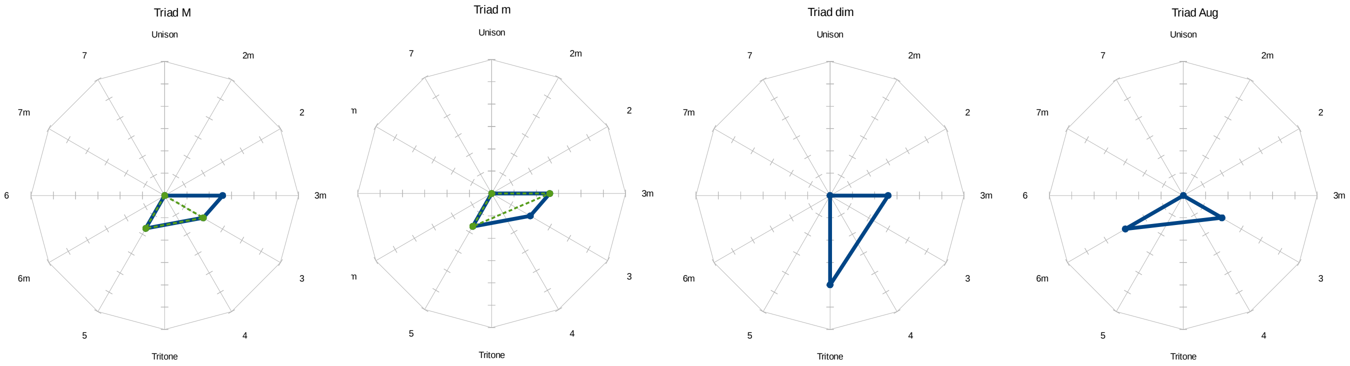




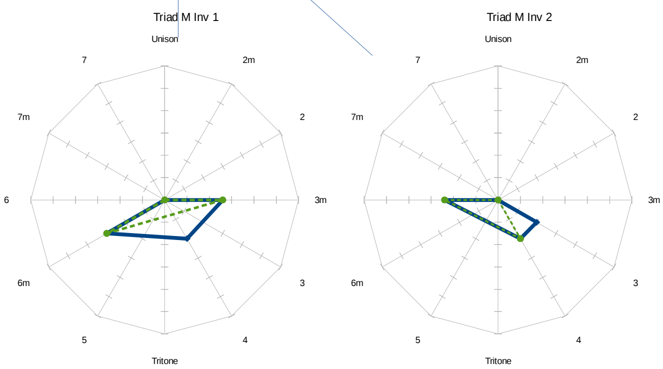
Dark

APPENDIX B: CHORDS DISSONANCE (sonorities)

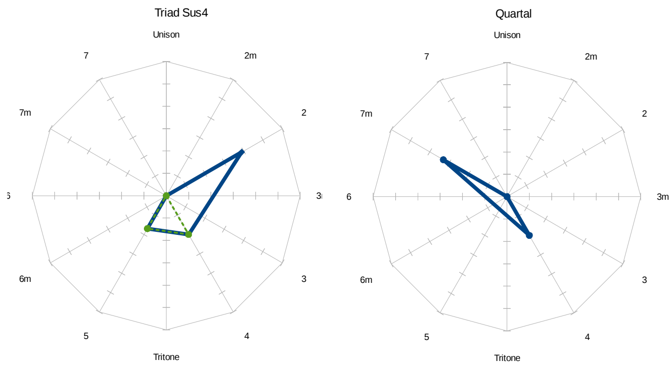
Triads



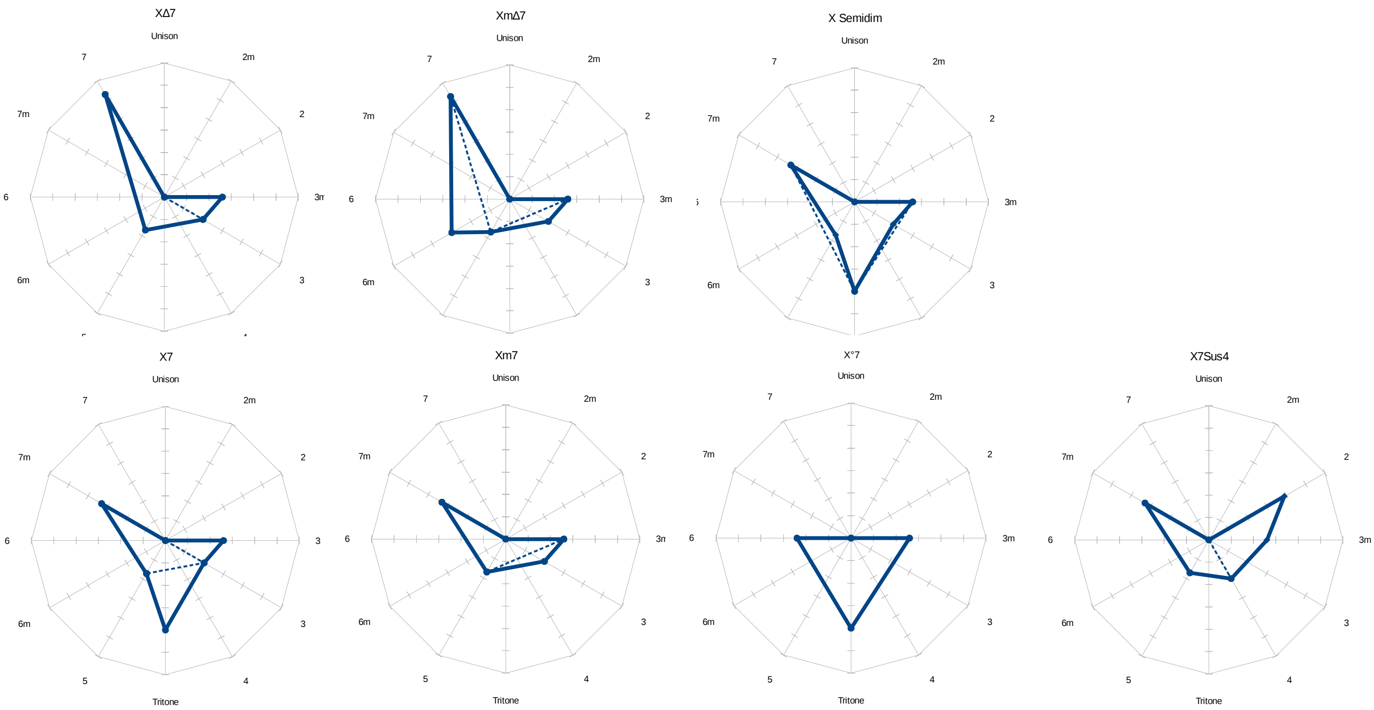
Triad inversions



Quartal chords

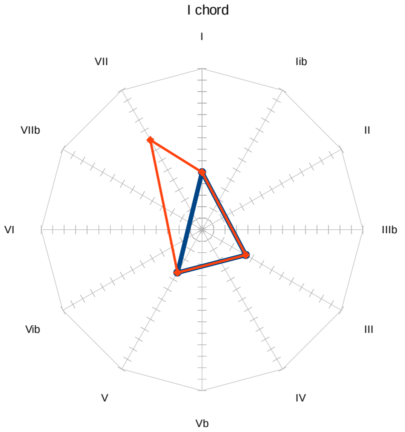


Triadic 7th chords

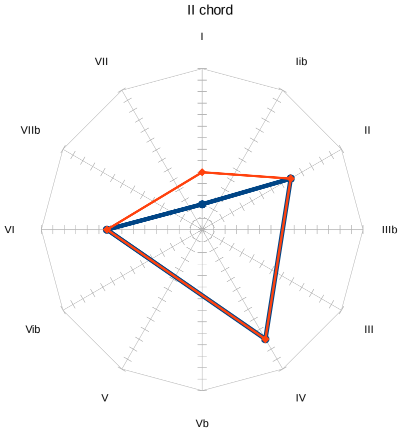


NB: Root position except for inversions. Dotted line plots represent extension with chord inner intervals.

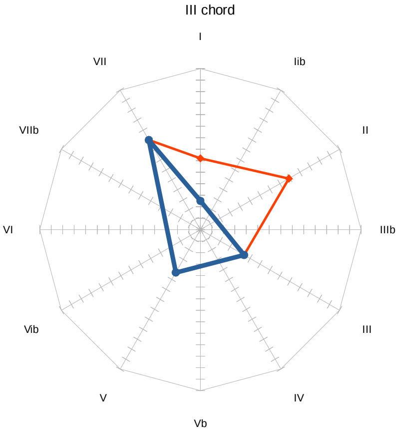
APPENDIX C: TENSION OF DIATONIC CHORDS IN MAJOR TONAL SPACE



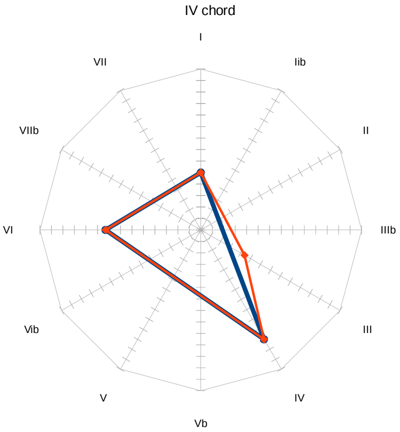
C, C$^{\Delta}$



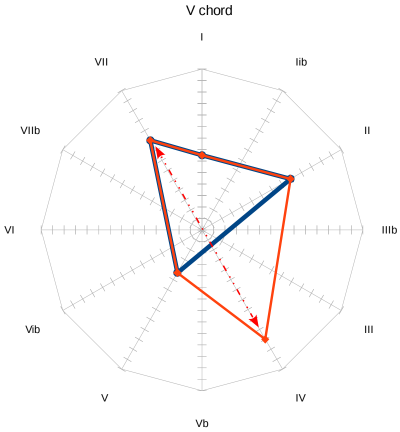
D^{m}, D^{m7}



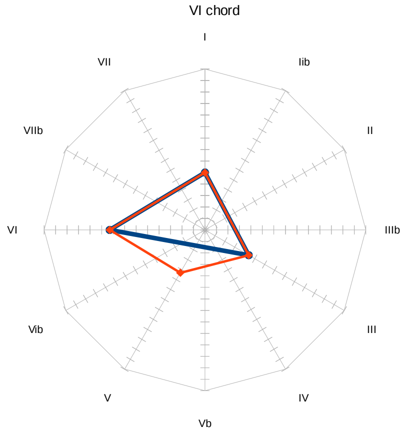
E^{m}, E^{m7}



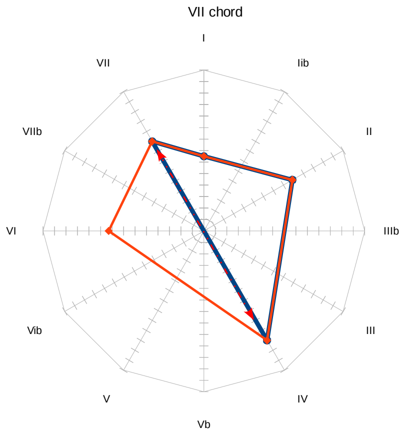
F, F$^{\Delta}$



G, G^{7}



A^{m}, A^{m7}



B^{ø}, B$^{\circ}$

NB:
Close position chords
In the diagrams the triadic and 7th chords (thin orange lines) superimposed.
Annotated chords for the C tonic.